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A Posteriori Error Estimates of Two-Grid Finite Element Methods for Nonlinear Elliptic Problems

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Abstract In this article, we study the residual-based a posteriori error estimates of the two-grid finite element methods for the second order nonlinear elliptic boundary value problems. Computable upper and lower bounds on the error in the H^1-norm are established. Numerical experiments are also provided to illustrate the performance of the proposed estimators.

Keywords Two-grid finite element method · Nonlinear elliptic problems · A posteriori error estimates

Mathematics Subject Classification 65N15 · 65N30

1 Introduction

The purpose of this article is to study the a posteriori error estimates of the two-grid finite element methods for the following second order nonlinear elliptic boundary value problems

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$$\begin{cases} \mathcal{L}u = -\nabla \cdot F(x, u, \nabla u) + g(x, u, \nabla u) = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a convex polygonal domain in $\mathbb{R}^2$ with the boundary $\partial\Omega$. We assume that $F(x, y, z) : \overline{\Omega} \times \mathbb{R}^1 \times \mathbb{R}^2 \to \mathbb{R}^2$ and $g(x, y, z) : \overline{\Omega} \times \mathbb{R}^1 \times \mathbb{R}^2 \to \mathbb{R}^1$ are smooth functions and that (1.1) has a solution $u \in H_0^1(\Omega) \cap W^{2,2+\epsilon}(\Omega)$ for some $\epsilon > 0$. The smoothness requirements on those functions will be given in detail later.

There are some important numerical results available for (1.1). Xu [43] proved the existence and uniqueness of the finite element approximations under the assumption $u \in H_0^1(\Omega) \cap W^{2,2+\epsilon}(\Omega)$ for some $\epsilon > 0$ and derived the optimal error estimates in the L^p- and $W^{1,p}$-norms. In particular, Xu [43] proposed some two-grid finite element algorithms to solve (1.1) and derived the convergence estimates to justify the efficiency of these algorithms. Verfürth [39] presented a general framework to derive the residual-based a posteriori error estimates in the H^1-norm for the finite element solutions of (1.1). Demlow [26] studied the residual-based pointwise a posteriori error estimates for gradients of piecewise linear finite element solutions of (1.1). Gudi et al. [28] and Bi et al. [17] analyzed the a priori and a posteriori error estimates of the hp-discontinuous Galerkin methods for (1.1), respectively. Bi and Ginting [15], Bi and Wang [16] studied the a priori and a posteriori error estimates of the finite volume element method for (1.1), respectively. Bi and Ginting [14] studied the global superconvergence and the postprocessing-based a posteriori error estimator of the finite element method for (1.1).

For a special equation of (1.1) with $F(x, u, \nabla u) = \alpha(x, u)\nabla u$ and $g(x, u, \nabla u) = g(x)$, we refer the reader to [21,27,32,33] for the finite element method, to [23,35] for the mixed finite element method, to [9–11,22] for the finite volume element method and to [12,13,29,30] for the discontinuous Galerkin methods.

Two-grid finite element methods, based on two finite element spaces S_H and S_h on one coarse grid H and one fine grid h, were first introduced by Xu [43–46] to solve the nonsymmetric linear and nonlinear elliptic problems. In the two-grid finite element methods, firstly, we use the standard finite element method to solve the nonlinear problems on the coarser space S_H and obtain a rough approximation $u_H \in S_H$. Secondly, based on u_H, we solve a linearized problem on the fine grid to get a corrected solution $u^h \in S_h$. A remarkable fact about this technique is that the space S_H can be extremely coarse, compared with S_h, to still maintain the optimal accuracy on the fine space. This means that solving a nonlinear equation is not much more difficult than solving a linear equation, since $\dim S_H \ll \dim S_h$ and the work for solving u_H is relatively very small.

Later on, the two-grid methods were further investigated by many authors, for instance, Xu and Zhou [46] for eigenvalue problems, Axelsson and Layton [4] for nonlinear elliptic problems, Dawson et al. [25] for the finite difference method for nonlinear parabolic equations, Utnes [38] for Navier–Stokes equations, Marion and Xu [34] for evolution equations, Wu and Allen [42] for the mixed finite element method to solve coupled reaction-diffusion systems, Bi and Ginting [10,12] and Bi et al. [18] for the finite volume element method and the discontinuous Galerkin finite element method for the nonlinear elliptic problems. Guo et al. [31] for the superconvergent two-grid methods for elliptic eigenvalue problems. Now the two-grid methods have been shown to be efficient techniques for solving nonlinear problems of various types.

A posteriori error estimates of the finite element method have been studied extensively in the past several decades and some important results have been achieved. We refer the reader to the monographs [2,6,8,36,40] and the references therein for an extensive survey of the vast amount of research in this field, many of which were concentrated on the linear problems. For the a posteriori error estimates of the nonlinear problems, we refer to [9,11,13,14,26,33,39,41] for details. Note that in those articles mentioned above, the higher regularity assumptions on the exact solutions of the nonlinear problems were required in the analysis of the a posteriori error estimates, which were used to guarantee the existence and uniqueness of the finite element approximations of the nonlinear problems.

In the application of the two-grid finite element methods to solve the nonlinear problem in practice, we require an accuracy verification of the numerical approximation. It is well known that the a posteriori error estimator allows us to monitor whether a numerical approximation is sufficiently accurate, even though the exact solution is unknown. This motivates us to study the a posteriori error estimates of the two-grid finite element methods.

To the best of our knowledge, the a posteriori error estimates of the two-grid finite element methods for the elliptic problems have not been studied previously. In this article, we study the a posteriori error estimates of two different two-grid finite element methods for a special case of (1.1) and (1.1), respectively. We propose two computationally easy residual-based a posteriori error estimators of the two-grid finite element methods. Under the assumption that the coarse grid of size H is sufficiently small, we develop the global upper and lower bounds on the error in the H^1-norm for the two-grid finite element methods. Note that the assumption that the mesh parameter is sufficiently small is reasonable, which guarantees the existence and uniqueness of the finite element approximation of (1.1), see [43] for details.

The organization of this article is as follows. In Sect. 2, we introduce some notation, formulate the finite element method and the two-grid finite element methods for a special case of (1.1) and (1.1), and give some lemmas used in the subsequent analysis. In Sects. 3 and 4, we propose the residual-based a posteriori error estimator of the two-grid finite element method for the special equation of (1.1) and (1.1), respectively, and derive the global upper and lower bounds on the error in the H^1-norm. Section 5 provides several numerical experiments to confirm our theoretical findings. Finally, in Sect. 6, we summarize the main results of this article and draw some conclusions.

2 Two-Grid Finite Element Methods

2.1 Nonlinear Elliptic Problems

Throughout this article, C denotes a generic positive constant, which may be dependent on the solutions of the nonlinear elliptic problems but is independent of the mesh parameter and may be different at different occurrences.

For integer $m \geq 0$ and real number $1 \leq p \leq \infty$, we employ the standard notation for the Sobolev spaces $W^{m,p}(\Omega)$, with the norm $\|\cdot\|_{m,p,\Omega}$ and the seminorm $|\cdot|_{m,p,\Omega}$ [1,19,24]. In order to simplify the notation, we denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ and skip the index $p = 2$ and Ω whenever possible, i.e., we will use $\|u\|_{m,p,\Omega} = \|u\|_{m,p}$, $\|u\|_{m,2,\Omega} = \|u\|_m$ and $\|u\|_0 = \|u\|$. The same convention is used for the seminorms as well. In addition, the space $H_0^1(\Omega)$ is defined, as usual, by $H_0^1(\Omega) = \{v \in H^1(\Omega) : \ v = 0 \ \text{ on } \ \partial\Omega\}$. In what follows, the symbol $|\cdot|$ will denote the area of a domain, and $(\cdot, \cdot)$ denote the $L^2(\Omega)$ inner product.

In this article, we admit the same assumptions on $F(x, y, z)$ and $g(x, y, z)$ as those in [43]. We assume that the functions $F_i(x, y, z)$, $i = 1, 2$, and $g(x, y, z)$, are twice continuously differentiable with respect to the variables y, z, and that (1.1) has a solution $u \in H_0^1(\Omega) \cap W^{2,2+\epsilon}(\Omega)$ for some $\epsilon > 0$. Moreover, for a given constant $C_1 > 0$, there exists a constant $M_1 > 0$ such that

$$\max_{x\in\overline{\Omega},|y|\le C_1,|z|\le C_1}\Big(|D_{yy}F|,|D_{yz}F|,|D_{zz}F|,|D_{yy}g|,|D_{yz}g|,|D_{zz}g|\Big)\le M_1. \tag{2.1}$$

Furthermore, for any $\omega\in W^{1,\infty}(\Omega)$, we denote

$$\begin{aligned}
a(\omega) &= D_zF(x,\omega,\nabla\omega)\in\mathbb{R}^{2\times 2}, \quad & b(\omega) &= D_yF(x,\omega,\nabla\omega)\in\mathbb{R}^2,\\
c(\omega) &= D_zg(x,\omega,\nabla\omega)\in\mathbb{R}^2, & d(\omega) &= D_yg(x,\omega,\nabla\omega)\in\mathbb{R}^1.
\end{aligned}$$

The linearized operator $\mathcal{L}$ at ω (namely the Fréchet derivative of $\mathcal{L}$ at ω) is then given by

$$\mathcal{L}'(\omega)v=-\nabla\cdot(a(\omega)\nabla v+b(\omega)v)+c(\omega)\cdot\nabla v+d(\omega)v.$$

Following [43], we maintain two basic assumptions to guarantee that u is an isolated solution of (1.1):

- For the solution of (1.1), $a(u)$ is a symmetric and uniformly positive definite (SPD) matrix in $\overline{\Omega}$, i.e.,
$$a_0|\xi|^2\le\xi^{\mathrm{T}}a(u)\xi,\quad \xi\in\mathbb{R}^2, x\in\overline{\Omega}, \tag{2.2}$$
for some constant $a_0>0$,
- $\mathcal{L}'(u): H_0^1(\Omega)\to H^{-1}(\Omega)$ is an isomorphism. (A simple sufficient condition for this assumption to be satisfied is $-\nabla\cdot b(u)+d(u)\ge 0$, see [43] for details).

For convenience of exposition, we introduce two parameters δ_2 and δ_1 as in [43]

$$\delta_2=\begin{cases}0, & \text{if } D_{zz}F(x,y,z)\equiv 0,\ D_{zz}g(x,y,z)\equiv 0\\ 1, & \text{otherwise}\end{cases}$$

and

$$\delta_1=\begin{cases}0, & \text{if } \delta_2=0,\ D_{yz}F(x,y,z)\equiv 0,\ D_{yz}g(x,y,z)\equiv 0\\ 1, & \text{otherwise.}\end{cases}$$

If $\delta_2=0$ and $\delta_1=1$, (1.1) is mildly nonlinear for which

$$\mathcal{L}u=-\nabla\cdot(\alpha(x,u)\nabla u+\beta(x,u))+\gamma(x,u)\cdot\nabla u+g(x,u).$$

If $\delta_2=\delta_1=0$, (1.1) is semilinear for which

$$\mathcal{L}u=-\nabla\cdot(\alpha(x)\nabla u+\beta(x,u))+g(x,u).$$

Problems (1.1) arise in several areas of applications. Keeping $u\in W^{2,2+\epsilon}(\Omega)$ in mind, from the Sobolev embedding inequality $||u||_{1,\infty}\le C||u||_{2,2+\epsilon}\le C_2$, we know that the following examples satisfy (2.1), (2.2) and can be treated with the technique presented in this article.

1. The stationary heat equation with convection and nonlinear diffusion coefficient:
$$F(x,u,\nabla u)=B(x,u)\nabla u,\quad g(x,u,\nabla u)=\beta(x)\cdot\nabla u+f(x),$$
where $B=(b_{ij})_{i,j=1}^2$ is a bounded uniformly positive definite matrix, $b_{ij}(x,y)$, $D_yb_{ij}(x,y)$ and $D_{yy}b_{ij}(x,y)$ are bounded and continuous on $\overline{\Omega}\times[-C_2,C_2]$ for all $i,j\in\{1,2\}$.
2. The equations of prescribed mean curvature:
$$F(x,u,\nabla u)=(1+|\nabla u|^2)^{-1/2}\nabla u,\quad g(x,u,\nabla u)=g(x).$$

3. Bratu's equation:

$$F(x, u, \nabla u) = \nabla u, \quad g(x, u, \nabla u) = \lambda e^u, \quad \lambda > 0.$$

4. A nonlinear eigenvalue problem:

$$F(x, u, \nabla u) = \nabla u, \quad g(x, u, \nabla u) = \lambda u - u^\kappa, \quad \lambda > 0, \quad \kappa \geq 2.$$

The weak formulation of (1.1) is

$$\text{Find } u \in H_0^1(\Omega) \text{ such that } A(u, v) = 0, \ \forall v \in H_0^1(\Omega), \tag{2.3}$$

where

$$A(u, v) = (F(u, \nabla u), \nabla v) + (g(u, \nabla u), v).$$

Here and hereafter, we do not specify the dependence of the functions F and g on x.

2.2 Finite Element Method

For the polygonal domain Ω, we consider a conforming triangulation $\mathcal{T}_h$ consisting of closed triangle element K such that $\overline{\Omega} = \cup_{K \in \mathcal{T}_h} K$, where $h = \max_{K \in \mathcal{T}_h}\{h_K\}$ and h_K is the diameter of the triangle K. Moreover, we assume that $\mathcal{T}_h$ is shape regular [19,24]. We denote by $\mathcal{E}_h^0$ the set of all interior edges of $\mathcal{T}_h$. To formulate the finite element approximation, conforming linear finite element space associated with $\mathcal{T}_h$ is defined as

$$S_h = \{v_h \in C(\overline{\Omega}) : v_h|_K \text{ is linear for all } K \in \mathcal{T}_h \text{ and } v_h|_{\partial\Omega} = 0\}.$$

Then the finite element approximation of (1.1) is

$$\text{Find } u_h \in S_h \text{ such that } A(u_h, v_h) = 0, \ \forall v_h \in S_h. \tag{2.4}$$

Existence and uniqueness of the solutions of (2.4) have been proved in [43].

Lemma 1 ([43]) *For sufficiently small h, the finite element Eq. (2.4) has a solution u_h satisfying*

$$\|u - u_h\|_{1,\infty} \leq Ch^\sigma \tag{2.5}$$

for some $\sigma > 0$. Furthermore there exists a constant $\mu > 0$ such that u_h is the only solution satisfying $\|u - u_h\|_{1,\infty} \leq \mu$.

Next, we state the error estimates of the finite element solution u_h in the $W^{1,p}$- and L^p-norms which have been developed in [43].

Lemma 2 ([43]) *Assume that $u \in W^{2,2+\epsilon}(\Omega)$, $\epsilon > 0$, and $u_h \in S_h$ are the solutions of (1.1) and (2.4), respectively, that satisfy (2.5). Then,*

$$\|u - u_h\|_{1,p} \leq Ch, \quad \text{if } u \in W^{2,p}(\Omega), \quad 2 \leq p \leq \infty,$$
$$\|u - u_h\|_{0,p} \leq Ch^2, \quad \text{if } u \in W^{2,p}(\Omega), \quad 2 \leq p < \infty,$$

for sufficiently small h.

We introduce the bilinear form $A'(\omega; \cdot, \cdot)$ induced by $\mathcal{L}'(\omega)$, for fixed ω,

$$A'(\omega; u, v) = (a(\omega)\nabla u + b(\omega)u, \nabla v) + (c(\omega) \cdot \nabla u + d(\omega)u, v). \tag{2.6}$$

The next lemma constructs a relationship between the bilinear form induced by $\mathcal{L}'(\omega)$ and $A(\cdot, \cdot)$, which serves as an auxiliary tool for later analysis.

Lemma 3 ([43]) *For any $v, v_h, \chi \in H_0^1(\Omega)$,*

$$A(v_h, \chi) = A(v, \chi) + A'(v; v_h - v, \chi) + R(v, v_h, \chi). \tag{2.7}$$

Thus $u_h \in S_h$ solves (2.4) if and only if

$$A'(u; u - u_h, \chi) = R(u, u_h, \chi), \quad \chi \in S_h.$$

Denoting $\varepsilon_h = v - v_h$, the remainder R satisfies

$$|R(v, v_h, \chi)| \le C(||\varepsilon_h||_{0,2p}^2 + \delta_2||\nabla\varepsilon_h||_{0,2p}^2 + \delta_1||\varepsilon_h\nabla\varepsilon_h||_{0,p})||\nabla\chi||_{0,q}, \tag{2.8}$$

for any v and v_h satisfying $||v||_{1,\infty} + ||v_h||_{1,\infty} \le M_2$, with given $M_2 > 0$, and C depends on M_2, and $1/p + 1/q = 1$, $p, q \ge 1$.

Since $\mathcal{L}'(u) : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isomorphism, we have the following result on the bilinear form $A'(u; \cdot, \cdot)$, see also [43], whose proof can be found in [5].

Lemma 4 *Assume that u is the solution of (1.1). For each $\omega \in H_0^1(\Omega)$, we have*

$$||\omega||_1 \le C \sup_{0\ne v\in H_0^1(\Omega)} \frac{A'(u; \omega, v)}{||v||_1}.$$

The following well-known trace theorem can be found in [3].

Lemma 5 ([3]) *For each $\omega \in H^1(K)$ and $E \in \partial K$, there exists a constant C independent of h_K such that*

$$||\omega||_{0,E}^2 \le C(h_K^{-1}||\omega||_{0,K}^2 + h_K||\nabla\omega||_{0,K}^2), \quad \forall K \in \mathcal{T}_h.$$

2.3 Two-Grid Finite Element Methods for Nonlinear Problems

In this subsection, we present two different two-grid finite element algorithms to solve the nonlinear elliptic problems as in [43]. The basic mechanism in these algorithms is to construct two shape-regular subdivisions of Ω, $\mathcal{T}_H$ and $\mathcal{T}_h$ with different mesh sizes H and $h (h \ll H)$.

The corresponding finite element spaces are S_H and S_h, which will be called coarse and fine space, respectively.

2.3.1 A Simple Two-Grid Method for Mildly Nonlinear Equations

We first present a simple two-grid finite element method for the following mildly nonlinear equations with homogeneous Dirichlet's boundary condition

$$\mathcal{L}u = -\nabla\cdot(\alpha(x, u)\nabla u + \beta(x, u)) + \gamma(x, u)\cdot\nabla u + g(x, u) = 0, \quad \text{in } \Omega. \tag{2.9}$$

This equation is a special case of (1.1) ($\delta_2 = 0$) with $F(x, y, z) = \alpha(x, y)z + \beta(x, y)$ and $g(x, y, z) = \gamma(x, y)\cdot z + g(x, y)$. We assume the early assumptions on (1.1) hold for (2.9). In particular, the functions $\alpha(x, y)$, $\beta(x, y)$, $\gamma(x, y)$ and $g(x, y)$ are twice continuously differentiable with respect to the second variable y. And for the solution of (2.9), $\alpha(x, u)$ is a symmetric and uniformly positive definite matrix in $\overline{\Omega}$, i.e.,

$$\alpha_0|\xi|^2 \le \xi^{\mathrm{T}}\alpha(x, u)\xi, \quad \xi \in \mathbb{R}^2, x \in \overline{\Omega}, \tag{2.10}$$

for some constant $\alpha_0 > 0$.

The weak formulation of (2.9) is

$$\text{Find } u \in H_0^1(\Omega) \text{ such that } \hat{A}(u, v) = 0, \ \forall v \in H_0^1(\Omega), \tag{2.11}$$

where

$$\hat{A}(u, v) = (\alpha(u)\nabla u + \beta(u), \nabla v) + (\gamma(u) \cdot \nabla u + g(u), v).$$

Here and hereafter, we do not specify the dependence of the functions α, β and γ on x. The finite element approximation of (2.9) is

$$\text{Find } u_h \in S_h \text{ such that } \hat{A}(u_h, v_h) = 0, \ \forall v_h \in S_h. \tag{2.12}$$

As a special case of (1.1), we know that Lemmas 1 and 2 also hold for (2.9).

Next, we present a two-grid finite element method, proposed in [43], for the mildly nonlinear Eq. (2.9), which reduces the nonlinear problem (2.12) to a SPD linear problem and a nonlinear system of smaller size. For this purpose, we define, for $\omega, u, v \in W^{1,\infty}(\Omega) \cap H_0^1(\Omega)$

$$\hat{A}_S(\omega; u, v) = (\alpha(\omega)\nabla u, \nabla v),$$

and

$$\hat{A}_N(\omega; u, v) = (\beta(\omega), \nabla v) + (\gamma(\omega) \cdot \nabla u + g(u), v).$$

The following Algorithm 1 is the precise statement of the two-grid finite element method for solving (2.9).

Algorithm 1 Two-grid finite element method for (2.9) [43]

1. Find $u_H \in S_H$ such that
$$\hat{A}(u_H, v_H) = 0, \quad \forall v_H \in S_H.$$
2. Find $u^h \in S_h$ such that
$$\hat{A}_S(u_H; u^h, v_h) + \hat{A}_N(u_H; u_H, v_h) = 0, \quad \forall v_h \in S_h.$$

The following error estimate has been developed in [43].

Lemma 6 ([43]) *Assume $u \in W^{2,2+\epsilon}(\Omega), \epsilon > 0$, and $u_h, u^h \in S_h$ are the solutions of (2.9), (2.12) and Algorithm 1, respectively, we have for $H \ll 1$*

$$||u_h - u^h||_1 \leq CH^2.$$

According to Lemmas 2 and 6, in order to obtain the optimal approximation for the solution of Algorithm 1, it suffices to take $H = \mathcal{O}(h^{1/2})$.

The lemma below gives a stability estimate of the two-grid finite element approximation u^h in the $W^{1,\infty}(\Omega)$-norm.

Lemma 7 *Let $u \in W^{2,2+\epsilon}(\Omega) \cap H_0^1(\Omega)$ and $u^h \in S_h$ be the solutions of (2.9) and Algorithm 1, respectively. Then, there exists a constant C independent of h such that for $H = \mathcal{O}(h^{\mu})$, $\mu \geq 1/2$, $\|u^h\|_{1,\infty} \leq C$.*

Proof Using the triangle inequality, Lemma 1, the inverse inequality [24], and Lemma 6, we have

$$\begin{aligned}||u^h||_{1,\infty} &\le ||u||_{1,\infty} + ||u-u_h||_{1,\infty} + ||u_h - u^h||_{1,\infty}\\ &\le ||u||_{1,\infty} + C + Ch^{-1}||u_h - u^h||_1\\ &\le ||u||_{1,\infty} + C + Ch^{-1}H^2.\end{aligned}$$

The desired result follows from the embedding theorem [19], $\|u\|_{1,\infty} \le C\|u\|_{2,2+\epsilon}$, and $H = \mathcal{O}(h^{\mu})$, $\mu \ge 1/2$. □

2.3.2 Correction by One Newton's Iteration on Fine Space

The following two-grid finite element method proposed in [43] applies to the general nonlinear Eq. (1.1). In this method, we use the coarse grid approximation as an initial guess for one Newton iteration on the fine grid.

Algorithm 2 Two-grid finite element method for (1.1) [43]

1. Find $u_H \in S_H$ such that
$$A(u_H, v_H) = 0, \quad \forall v_H \in S_H.$$
2. Find $u^h \in S_h$ such that
$$A'(u_H; u^h, v_h) = A'(u_H; u_H, v_h) - A(u_H, v_h), \quad \forall v_h \in S_h.$$

Next we state the error estimates in the H^1-norm and in the $W^{1,\infty}$-norm for the two-grid finite element method for (1.1) developed in [43].

Lemma 8 ([43]) *Assume u_h, $u^h \in S_h$ are the solutions of* (2.4) *and Algorithm* 2*, respectively, we have for $H \ll 1$*

$$\begin{aligned}&||u_h - u^h||_1 \le C(H^4 + \delta_1 H^3 + \delta_2 H^2), \quad u \in W^{2,4}(\Omega),\\ &||u_h - u^h||_{1,\infty} \le C(H^4 + \delta_1 H^3 + \delta_2 H^2)|\ln h|, \quad u \in W^{2,\infty}(\Omega).\end{aligned}$$

Remark 1 From Lemmas 2 and 8, we know that the optimal rate of $||u-u^h||_1$ for the two-grid finite element method for (1.1) can be achieved by employing, respectively, $H = \mathcal{O}(h^{1/2})$ for $\delta_2 = 1$, $H = \mathcal{O}(h^{1/3})$ for $\delta_2 = 0$, $\delta_1 = 1$ and $H = \mathcal{O}(h^{1/4})$ for $\delta_2 = \delta_1 = 0$.

The following error estimate in the $W^{1,4}$-norm for the two-grid finite element method for (1.1) will be used in the a posteriori error analysis.

Lemma 9 *Assume $u \in W^{2,4}(\Omega)$ and $u^h \in S_h$ are the solutions of* (1.1) *and Algorithm* 2*, respectively, we have for $H \ll 1$*

$$||u - u^h||_{1,4} \le Ch + Ch^{-\frac{1}{2}}(H^4 + \delta_1 H^3 + \delta_2 H^2).$$

Proof By the triangle inequality, the inverse inequality [24], Lemmas 2 and 8, we get

$$\begin{aligned}||u - u^h||_{1,4} &\le ||u-u_h||_{1,4} + ||u_h - u^h||_{1,4}\\ &\le Ch + Ch^{-\frac{1}{2}}||u_h - u^h||_1\\ &\le Ch + Ch^{-\frac{1}{2}}(H^4 + \delta_1 H^3 + \delta_2 H^2),\end{aligned}$$

which completes the proof. □

3 A Posteriori Error Estimates of Algorithm 1

In this section, we propose the residual-based a posteriori error estimator of Algorithm 1 for (2.9) and derive the computable upper and lower bounds on the error $u - u^h$ in the H^1-norm.

3.1 A Reliable Bound on the Error of Algorithm 1

In this subsection, we develop the computable upper bound on the error $u-u^h$ in the H^1-norm of Algorithm 1 for (2.9).

We first introduce the jump of a vector-valued function across the interior edge, which will be used in the definition of the residual-based a posteriori error estimator. Let E be an interior edge shared by elements K^+ and $K^-$and q be a vector-valued function, that is smooth inside each element $K^\pm$. $q^\pm$ denote the traces of q on E taken from within the interior of $K^\pm$, respectively. Then, the jump of q on E is given by $[\![q]\!] = q^+ \cdot \mathbf{n}_{K^+} + q^- \cdot \mathbf{n}_{K^-}$, where $\mathbf{n}_{K^\pm}$ denote the unit outward normal vector of $\partial K^\pm$, respectively.

The following lemma states some estimates for the Scott-Zhang interpolation function which preserves homogeneous boundary condition.

Lemma 10 ([37]) *For each $v \in H_0^1(\Omega)$, there exists a function $v^I \in S_h$ such that for any $K \in \mathcal{T}_h$, $E \in \mathcal{E}_h$,*

$$||v^I||_1 \le C||v||_1, \quad ||v - v^I||_{0,K} \le Ch_K||v||_{1,\omega_K}, \quad ||v - v^I||_{0,E} \le Ch_E^{\frac{1}{2}}||v||_{1,\omega_E},$$

where $\omega_K = \cup_{K' \cap K \neq \emptyset} K'$ and $\omega_E = \cup_{K \cap E \neq \emptyset} K$.

The following lemma plays a key role in the a posteriori error estimation.

Lemma 11 *Assume that $u \in W^{2,2+\epsilon}(\Omega)$, $\epsilon > 0$, and $u^h \in S_h$ are the solutions of (2.9) and Algorithm 1, respectively. Then for the error $u - u^h$ and $v \in H_0^1(\Omega)$, we have*

$$(\alpha(u)\nabla(u^h - u), \nabla v) = I_1 + \cdots + I_6, \tag{3.1}$$

where

$$\begin{aligned}
I_1 &= \sum_{K \in \mathcal{T}_h} \int_K \Big(-\nabla \cdot (\alpha(u_H)\nabla u^h + \beta(u_H)) + \gamma(u_H) \cdot \nabla u_H + g(u_H)\Big)(v - v^I)\mathrm{d}x,\\
I_2 &= \sum_{E \in \mathcal{E}_h^0} \int_E [\![\alpha(u_H)\nabla u^h]\!](v - v^I)\mathrm{d}s, \quad I_3 = ((\alpha(u) - \alpha(u_H))\nabla u^h, \nabla v),\\
I_4 &= (\beta(u) - \beta(u_H), \nabla v), \quad I_5 = (\gamma(u) \cdot \nabla u - \gamma(u_H) \cdot \nabla u_H, v),\\
I_6 &= (g(u) - g(u_H), v),
\end{aligned}$$

and v^I is the Scott-Zhang interpolant of v given in Lemma 10.

Proof With the help of (2.11), we write $(\alpha(u)\nabla(u^h - u), \nabla v)$ as follows

$$\begin{aligned}
&(\alpha(u)\nabla(u^h - u), \nabla v)\\
&\quad = (\alpha(u)\nabla u^h, \nabla v) - (\alpha(u)\nabla u, \nabla v)\\
&\quad = (\alpha(u)\nabla u^h, \nabla v) + (\beta(u), \nabla v) + (\gamma(u)\cdot\nabla u, v) + (g(u), v)\\
&\quad = (\alpha(u)\nabla u^h, \nabla(v - v^I)) + (\beta(u), \nabla(v - v^I))\\
&\qquad + (\gamma(u)\cdot\nabla u, v - v^I) + (g(u), v - v^I) + (\alpha(u)\nabla u^h, \nabla v^I)\\
&\qquad + (\beta(u), \nabla v^I) + (\gamma(u)\cdot\nabla u, v^I) + (g(u), v^I)\\
&\quad = (\alpha(u_H)\nabla u^h, \nabla(v - v^I)) + (\beta(u_H), \nabla(v - v^I))\\
&\qquad + (\gamma(u_H)\cdot\nabla u_H, v - v^I) + (g(u_H), v - v^I)\\
&\qquad + ((\alpha(u) - \alpha(u_H))\nabla u^h, \nabla(v - v^I)) + (\beta(u) - \beta(u_H), \nabla(v - v^I))\\
&\qquad + (\gamma(u)\cdot\nabla u - \gamma(u_H)\cdot\nabla u_H, v - v^I) + (g(u) - g(u_H), v - v^I)\\
&\qquad + (\alpha(u)\nabla u^h, \nabla v^I) + (\beta(u), \nabla v^I) + (\gamma(u)\cdot\nabla u, v^I) + (g(u), v^I).
\end{aligned} \tag{3.2}$$

Since $v - v^I \in H_0^1(\Omega)$, applying Green's formula to the first and the second terms on the right-hand side of (3.2) gives

$$\begin{aligned}
(\alpha(u_H)\nabla u^h, \nabla(v - v^I)) = &-\sum_{K\in\mathcal{T}_h}\int_K \nabla\cdot(\alpha(u_H)\nabla u^h)(v - v^I)\mathrm{d}x\\
&+\sum_{E\in\mathcal{E}_h^0}\int_E [\![\alpha(u_H)\nabla u^h]\!](v - v^I)\mathrm{d}s.
\end{aligned} \tag{3.3}$$

and

$$(\beta(u_H), \nabla(v - v^I)) = -(\nabla\cdot\beta(u_H), v - v^I). \tag{3.4}$$

From Algorithm 1, we know that

$$(\alpha(u_H)\nabla u^h, \nabla v^I) + (\beta(u_H), \nabla v^I) + (\gamma(u_H)\cdot\nabla u_H + g(u_H), v^I) = 0. \tag{3.5}$$

Then, substituting (3.3), (3.4) into (3.2) and using (3.5) yield

$$\begin{aligned}
&(\alpha(u)\nabla(u^h - u), \nabla v)\\
&= \sum_{K\in\mathcal{T}_h}\int_K \Big(-\nabla\cdot(\alpha(u_H)\nabla u^h + \beta(u_H)) + \gamma(u_H)\cdot\nabla u_H + g(u_H)\Big)(v - v^I)\mathrm{d}x\\
&\quad + \sum_{E\in\mathcal{E}_h^0}\int_E [\![\alpha(u_H)\nabla u^h]\!](v - v^I)\mathrm{d}s + ((\alpha(u) - \alpha(u_H))\nabla u^h, \nabla v)\\
&\quad + (\beta(u) - \beta(u_H), \nabla v) + (\gamma(u)\cdot\nabla u - \gamma(u_H)\cdot\nabla u_H, v)\\
&\quad + (g(u) - g(u_H), v),
\end{aligned}$$

which completes the proof. □

Motivated by the above lemma, we introduce locally computable quantities which will be used in the definition of the residual-based a posteriori error estimator.

Definition 1 On each element $K \in \mathcal{T}_h$ and each interior edge $E \in \mathcal{E}_h^0$, define the element residual and the edge jump by, respectively,

$$\begin{aligned} R_{1,K} &= R_{1,K}(u^h) = -\nabla \cdot (\alpha(u_H)\nabla u^h + \beta(u_H)) + \gamma(u_H) \cdot \nabla u_H + g(u_H), \\ J_{1,E} &= J_{1,E}(u^h) = [\![\alpha(u_H)\nabla u^h]\!]_E \end{aligned}$$

and define the local error estimators $\eta_{1,R}(K)$ and $\eta_{1,J}(E)$ by

$$\eta_{1,R}(K)^2 = h_K^2 ||R_{1,K}||_{0,K}^2, \quad \eta_{1,J}(E)^2 = h_E ||J_{1,E}||_{0,E}^2.$$

Define the global error estimators by

$$\eta_{1,R} = \left(\sum_{K \in \mathcal{T}_h} \eta_{1,R}(K)^2 \right)^{\frac{1}{2}}, \quad \eta_{1,J} = \left(\sum_{E \in \mathcal{E}_h^0} \eta_{1,J}(E)^2 \right)^{\frac{1}{2}}.$$

In order to simplify the notation, we shall use the concept of "higher order term" (*h.o.t.*) as in [20]. From Lemmas 2 and 6, we know that $||u - u^h||_1 \le Ch$ for $H = \mathcal{O}(h^\mu)$ with $\mu \ge 1/2$. It is reasonable to denote the term that tends to zero faster than h by *h.o.t.*.

We are now in position to develop a reliable estimate up to a higher order term (*h.o.t.*) for the error $u - u^h$ in the H^1-norm of Algorithm 1 for (2.9). For this purpose, we will estimate the terms on the right-hand side of (3.1) separately.

Theorem 1 *Assume that* $u \in W^{2,2+\epsilon}(\Omega)$, $\epsilon > 0$ *and* $u^h \in S_h$ *are the solutions of* (2.9) *and Algorithm* 1*, respectively. Then, we can choose* $H = \mathcal{O}(h^\mu)$, $\mu > 1/2$ *such that*

$$||u - u^h||_1 \le C(\eta_{1,R} + \eta_{1,J}) + h.o.t..$$

Proof It follows from Cauchy–Schwarz inequality and Lemma 10

$$\begin{aligned} |I_1| &= \left| \sum_{K \in \mathcal{T}_h} \int_K R_{1,K}(v - v^I) \mathrm{d}x \right| \\ &\le \left(\sum_{K \in \mathcal{T}_h} h_K^2 \| R_{1,K} ||_{0,K}^2 \right)^{\frac{1}{2}} \left(\sum_{K \in \mathcal{T}_h} h_K^{-2} ||v - v^I||_{0,K}^2 \right)^{\frac{1}{2}} \\ &\le C\eta_{1,R} ||v||_1. \end{aligned} \tag{3.6}$$

The estimation of I_2 is obtained by using Cauchy–Schwarz inequality, Lemmas 5 and 10 in the following calculation

$$\begin{aligned} |I_2| &= \left| \sum_{E \in \mathcal{E}_h^0} \int_E J_{1,E}(v - v^I) \mathrm{d}s \right| \\ &\le \sum_{E \in \mathcal{E}_h^0} ||J_{1,E}||_{0,E} ||v - v^I||_{0,E} \\ &\le \left(\sum_{E \in \mathcal{E}_h^0} h_E ||J_{1,E}||_{0,E}^2 \right)^{\frac{1}{2}} \left(\sum_{E \in \mathcal{E}_h^0} h_E^{-1} ||v - v^I||_{0,E}^2 \right)^{\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
&\leq C\eta_{1,J}\left(\sum_{K\in\mathcal{T}_h}\left(h_K^{-2}||v-v^I||_{0,K}^2+||v-v^I||_{1,K}^2\right)\right)^{\frac{1}{2}}\\
&\leq C\eta_{1,J}||v||_1. \qquad (3.7)
\end{aligned}
$$

From Cauchy–Schwarz inequality, Lemmas 7 and 2, we have

$$
|I_3|\leq C||u^h||_{1,\infty}||u-u_H||\;||v||_1\leq CH^2||v||_1. \qquad (3.8)
$$

Similarly, we can get the estimation of I_4 and I_6

$$
|I_4|+|I_6|\leq C||u-u_H||\;||v||_1\leq CH^2||v||_1. \qquad (3.9)
$$

For the term I_5, from the triangle inequality, we have

$$
|I_5|\leq|(\gamma(u)\cdot\nabla(u-u_H),v)|+|((\gamma(u)-\gamma(u_H))\cdot\nabla u_H,v)|. \qquad (3.10)
$$

From Green's formula, $|\gamma'(y)|\leq C$ and $|u|_{1,\infty}\leq C||u||_{2,2+\epsilon}\leq C$, we get

$$
\begin{aligned}
|(\gamma(u)\cdot\nabla(u-u_H),v)|&=|(\nabla(u-u_H),\gamma(u)v)|=|(u-u_H,\nabla(\gamma(u)v))|\\
&\leq||u-u_H||\;|\gamma(u)v|_1\leq||u-u_H||\;|\gamma(u)|_{1,\infty}|v|_1\\
&\leq C||u-u_H||\;|v|_1\leq CH^2|v|_1. \qquad (3.11)
\end{aligned}
$$

Since $||u_H||_{1,\infty}\leq C$, the estimation of the second term on the right-hand side of (3.10) is similar to I_3 in (3.8). Then

$$
|I_5|\leq CH^2||v||_1. \qquad (3.12)
$$

Letting $v=u^h-u$ in (3.1). Then from (2.10), the equivalence of $||\cdot||_1$ and $|\cdot|_1$ in $H_0^1(\Omega)$, (3.6)–(3.9) and (3.12), we have

$$
||u-u^h||_1\leq C(\eta_{1,R}+\eta_{1,J})+CH^2.
$$

The desired result follows since H^2 is a *h.o.t* for $H=\mathcal{O}(h^\mu)$, $\mu>1/2$. □

3.2 A Lower Bound on the Error of Algorithm 1

In this subsection, we derive the lower bound on the error $u-u^h$ in the H^1-norm of Algorithm 1 for (2.9). For this purpose, we introduce the oscillations of the residual $R_{1,K}$ and the jump $J_{1,E}$ over the element K and the interior edge E

$$
\begin{aligned}
\mathrm{osc}_{1,R}(K)&=h_K\|R_{1,K}-\overline{R}_{1,K}\|_{0,K}\quad\text{and}\\
\mathrm{osc}_{1,J}(E)&=h_E^{\frac{1}{2}}\|J_{1,E}-\overline{J}_{1,E}\|_{0,E},
\end{aligned}
$$

where $\overline{R}_{1,K}$ is the average of $R_{1,K}$ over K and $\overline{J}_{1,E}$ is the average of $J_{1,E}$ on E, which are respectively defined as

$$
\overline{R}_{1,K}=\frac{1}{|K|}\int_K R_{1,K}\mathrm{d}x,\quad \overline{J}_{1,E}=\frac{1}{h_E}\int_E J_{1,E}\mathrm{d}s.
$$

We denote the total oscillations by

$$
\mathrm{osc}_1=\left(\sum_{K\in\mathcal{T}_h}\mathrm{osc}_1(K)^2\right)^{\frac{1}{2}},
$$

where

$$\mathrm{osc}_1(K)^2 = \mathrm{osc}_{1,R}(K)^2 + \sum_{E\in\partial K} \mathrm{osc}_{1,J}(E)^2.$$

Remark 2 From Lemmas 2 and 6, we know that $\|u-u^h\|_1 \le Ch$ for $H = \mathcal{O}(h^\mu)$, $\mu \ge 1/2$. In contrast to this estimate, osc_1, $\mathrm{osc}_{1,R}$ tend to zero faster than $\mathcal{O}(h)$ and are also higher order terms, see p. 2506 in [20] for details.

In order to localize the error estimate on a triangle $K \in \mathcal{T}_h$, we employ the bubble function as in [40]

$$b_K = \begin{cases} 27\lambda_{K,1}\lambda_{K,2}\lambda_{K,3}, & \text{on} \quad K, \\ 0, & \text{on} \quad \Omega\backslash K. \end{cases}$$

where $\lambda_{K,1}, \lambda_{K,2}, \lambda_{K,3}$ denote the barycentric coordinates associated with K. Furthermore, given two triangles K and K' (such that $\omega_E = K \cup K'$) sharing an interior edge E, we set

$$b_E = \begin{cases} 4\lambda_{K,k}\lambda_{K,j}, & \text{on} \quad K, \\ 4\lambda_{K',l}\lambda_{K',m}, & \text{on} \quad K', \\ 0, & \text{on} \quad \Omega\backslash\omega_E, \end{cases}$$

$\lambda_{K,k}$ and $\lambda_{K,j}$ are the barycentric coordinates associated with K with $\lambda_{K,k}\lambda_{K,j} = 0$ on $\partial K\backslash E$. The functions $\lambda_{K',l}$ and $\lambda_{K',m}$ are such that $\lambda_{K',l}\lambda_{K',n} = 0$ on $\partial K'\backslash E$. These bubble functions possess the following properties.

Lemma 12 ([40]) *The functions b_K and b_E satisfy the following properties:*

1. $\mathrm{supp}(b_K) \subset K$, $b_K \in [0,1]$, *and* $\max_{x\in K} b_K(x) = 1$;

$$\int_K b_K \mathrm{d}x = \frac{9}{20}|K| \sim h_K^2, \quad \|\nabla b_K\|_{0,K} \le Ch_K^{-1}\|b_K\|_{0,K};$$

2. $\mathrm{supp}(b_E) \subset \omega_E$, $b_E \in [0,1]$, *and* $\max_{x\in\omega_E} b_E(x) = 1$; $\int_E b_E \mathrm{d}s = \frac{2}{3}h_E$,

$$\int_{\omega_E} b_E \mathrm{d}x = \frac{1}{3}|\omega_E| \sim h_E^2; \quad \|\nabla b_E\|_{0,\omega_E} \le Ch_E^{-1}\|b_E\|_{0,\omega_E}.$$

From Lemma 12, we get the following estimates which will be used in the subsequent analysis

$$\begin{aligned} ||\nabla(b_K\overline{R}_{1,K})||_{0,K} &= ||\nabla b_K||_{0,K}|\overline{R}_{1,K}| \le Ch_K^{-1}||b_K||_{0,K}|\overline{R}_{1,K}| \\ &\le Ch_K^{-1}||b_K\overline{R}_{1,K}||_{0,K} \le Ch_K^{-1}||\overline{R}_{1,K}||_{0,K}, \end{aligned} \tag{3.13}$$

$$||b_E\overline{J}_{1,E}||_{0,\omega_E} = ||b_E||_{0,\omega_E}|\overline{J}_{1,E}| \le Ch_E|\overline{J}_{1,E}| \le Ch_E^{\frac{1}{2}}||\overline{J}_{1,E}||_{0,E}, \tag{3.14}$$

and

$$\begin{aligned} ||\nabla(b_E\overline{J}_{1,E})||_{0,\omega_E} &= ||\nabla b_E||_{0,\omega_E}|\overline{J}_{1,E}| \le Ch_E^{-1}||b_E||_{0,\omega_E}|\overline{J}_{1,E}| \\ &= Ch_E^{-1}||b_E\overline{J}_{1,E}||_{0,\omega_E} \le Ch_E^{-\frac{1}{2}}||\overline{J}_{1,E}||_{0,E}, \end{aligned} \tag{3.15}$$

where (3.14) is used in the last inequality in (3.15).

First, for the element residual $R_{1,K}$, we have the following lower bound.

Lemma 13 *Assume that* $u \in W^{2,2+\epsilon}(\Omega)$, $\epsilon > 0$, *and* $u^h \in S_h$ *are the solutions of* (2.9) *and Algorithm* 1*, respectively. Then, there exists a constant* $C > 0$ *such that*

$$\eta_{1,R}(K)^2 \leq C||u-u^h||_{1,K}^2 + C||u-u_H||_{0,K}^2 + C\mathrm{osc}_{1,R}(K)^2, \quad \forall K \in \mathcal{T}_h.$$

Proof By the triangle inequality,

$$h_K||R_{1,K}||_{0,K} \leq h_K||\overline{R}_{1,K}||_{0,K} + \mathrm{osc}_{1,R}(K). \tag{3.16}$$

Thus, we only estimate $h_K||\overline{R}_{1,K}||_{0,K}$ in the following. By the properties of b_K in Lemma 12, the definition of $R_{1,K}$, Green's formula and

$$(\alpha(u)\nabla u + \beta(u), \nabla(b_K\overline{R}_{1,K})) + (\gamma(u)\cdot\nabla u + g(u), b_K\overline{R}_{1,K}) = 0$$

we get

$$\begin{aligned}
\frac{9}{20}||\overline{R}_{1,K}||_{0,K}^2 &= (\overline{R}_{1,K}, b_K\overline{R}_{1,K}) \\
&= (R_{1,K}, b_K\overline{R}_{1,K}) - (R_{1,K}-\overline{R}_{1,K}, b_K\overline{R}_{1,K}) \\
&= (-\nabla\cdot(\alpha(u_H)\nabla u^h + \beta(u_H)) + \gamma(u_H)\cdot\nabla u_H + g(u_H), b_K\overline{R}_{1,K}) \\
&\quad - (R_{1,K}-\overline{R}_{1,K}, b_K\overline{R}_{1,K}) \\
&= (\alpha(u_H)\nabla u^h + \beta(u_H), \nabla(b_K\overline{R}_{1,K})) \\
&\quad + (\gamma(u_H)\cdot\nabla u_H + g(u_H), b_K\overline{R}_{1,K}) - (R_{1,K}-\overline{R}_{1,K}, b_K\overline{R}_{1,K}) \\
&= (\alpha(u_H)\nabla u^h - \alpha(u)\nabla u, \nabla(b_K\overline{R}_{1,K})) \\
&\quad + (\beta(u_H)-\beta(u), \nabla(b_K\overline{R}_{1,K})) \\
&\quad + (\gamma(u_H)\cdot\nabla u_H - \gamma(u)\cdot\nabla u, b_K\overline{R}_{1,K}) \\
&\quad + (g(u_H)-g(u), b_K\overline{R}_{1,K}) - (R_{1,K}-\overline{R}_{1,K}, b_K\overline{R}_{1,K}) \\
&= S_1 + S_2 + S_3 + S_4 + S_5.
\end{aligned} \tag{3.17}$$

From Cauchy–Schwarz inequality, Lemma 7 and (3.13), we have

$$\begin{aligned}
|S_1| &\leq |((\alpha(u_H)-\alpha(u))\nabla u^h, \nabla(b_K\overline{R}_{1,K}))| + |(\alpha(u)\nabla(u^h-u), \nabla(b_K\overline{R}_{1,K}))| \\
&\leq C|u^h|_{1,\infty}||u-u_H||_{0,K}||\nabla(b_K\overline{R}_{1,K})||_{0,K} + C|u-u^h|_{1,K}||\nabla(b_K\overline{R}_{1,K})||_{0,K} \\
&\leq Ch_K^{-1}||u-u_H||_{0,K}||\overline{R}_{1,K}||_{0,K} + Ch_K^{-1}|u-u^h|_{1,K}||\overline{R}_{1,K}||_{0,K}.
\end{aligned} \tag{3.18}$$

Keeping (3.13) in mind, using the method to estimate I_4, I_5 and I_6 in the proof of Theorem 1, we have

$$|S_2| + |S_3| + |S_4| \leq Ch_K^{-1}||u-u_H||_{0,K}||\overline{R}_{1,K}||_{0,K}. \tag{3.19}$$

Using Cauchy–Schwarz inequality and $\max_{x\in K} b(x) = 1$, we get

$$|S_5| \leq ||R_{1,K}-\overline{R}_{1,K}||_{0,K}||b_K\overline{R}_{1,K}||_{0,K} \leq ||R_{1,K}-\overline{R}_{1,K}||_{0,K}||\overline{R}_{1,K}||_{0,K}. \tag{3.20}$$

Combining (3.18), (3.19), (3.20) with (3.17) yields

$$h_K||\overline{R}_{1,K}||_{0,K} \leq C||u-u^h||_{1,K} + C||u-u_H||_{0,K} + Ch_K||R_K - \overline{R}_{1,K}||_{0,K}.$$

The desired result follows from (3.16) and the above inequality. □

Lemma 14 *Assume that* $u \in W^{2,2+\epsilon}(\Omega)$, $\epsilon > 0$, *and* $u^h \in S_h$ *are the solutions of* (2.9) *and Algorithm* 1*, respectively. Then, there exists a constant* $C > 0$ *such that for* $E = \partial K \cap \partial K'$

$$\eta_{1,J}(E)^2 \le C||u-u^h||_{1,\omega_E}^2 + C||u-u_H||_{0,\omega_E}^2 + C\mathrm{osc}_{1,R}(\omega_E)^2 + C\mathrm{osc}_{1,J}(E)^2,$$

where $\omega_E = K \cup K'$, $\mathrm{osc}_{1,R}(\omega_E)^2 = \mathrm{osc}_{1,R}(K)^2 + \mathrm{osc}_{1,R}(K')^2$.

Proof Using the triangle inequality once more,

$$h_E^{\frac{1}{2}}||J_{1,E}||_{0,E} \le h_E^{\frac{1}{2}}||\overline{J}_{1,E}||_{0,E} + \mathrm{osc}_{1,J}(E). \tag{3.21}$$

Next, we only estimate $h_E^{\frac{1}{2}}||\overline{J}_{1,E}||_{0,E}$. By the properties of b_E in Lemma 12,

$$\frac{2}{3}||\overline{J}_{1,E}||_{0,E}^2 = (J_{1,E}, b_E\overline{J}_{1,E})_E + (\overline{J}_{1,E} - J_{1,E}, b_E\overline{J}_{1,E})_E. \tag{3.22}$$

Using Cauchy–Schwarz inequality and $\max_{x\in\omega_E} b_E(x) = 1$ to get

$$|(\overline{J}_{1,E} - J_{1,E}, b_E\overline{J}_{1,E})_E| \le ||\overline{J}_{1,E} - J_{1,E}||_{0,E}||\overline{J}_{1,E}||_{0,E}. \tag{3.23}$$

Since $b_E\overline{J}_{1,E} \in H_0^1(\omega_E) \subset H_0^1(\Omega)$, we have

$$(\alpha(u)\nabla u + \beta(u), \nabla(b_E\overline{J}_{1,E}))_{\omega_E} + (\gamma(u) + g(u), b_E\overline{J}_{1,E})_{\omega_E} = 0. \tag{3.24}$$

Using $[\![\beta(u_H)]\!]_E = 0$, the definition of $J_{1,E}$, Green's formula and (3.24), we get

$$\begin{aligned}
&(J_{1,E}, b_E\overline{J}_{1,E})_E \\
&\quad = ([\![\alpha(u_H)\nabla u^h]\!], b_E\overline{J}_{1,E})_E \\
&\quad = ([\![\alpha(u_H)\nabla u^h + \beta(u_H)]\!], b_E\overline{J}_{1,E})_E \\
&\quad = (\alpha(u_H)\nabla u^h + \beta(u_H), \nabla(b_E\overline{J}_{1,E}))_{\omega_E} \\
&\qquad + (\nabla_h\cdot(\alpha(u_H)\nabla u^h + \beta(u_H)), b_E\overline{J}_{1,E})_{\omega_E} \\
&\quad = (\alpha(u_H)\nabla u^h + \beta(u_H), \nabla(b_E\overline{J}_{1,E}))_{\omega_E} + (\gamma(u_H)\cdot\nabla u_H + g(u_H), b_E\overline{J}_{1,E})_{\omega_E} \\
&\qquad + (\nabla_h\cdot(\alpha(u_H)\nabla u^h + \beta(u_H)) - \gamma(u_H)\cdot\nabla u_H - g(u_H), b_E\overline{J}_{1,E})_{\omega_E} \\
&\quad = (\alpha(u_H)\nabla u^h - \alpha(u)\nabla u, \nabla(b_E\overline{J}_{1,E}))_{\omega_E} + (\beta(u_H) - \beta(u), \nabla(b_E\overline{J}_{1,E}))_{\omega_E} \\
&\qquad + (\gamma(u_H)\cdot\nabla u_H - \gamma(u)\cdot\nabla u, b_E\overline{J}_{1,E})_{\omega_E} + (g(u_H) - g(u), b_E\overline{J}_{1,E})_{\omega_E} \\
&\qquad + (\nabla_h\cdot(\alpha(u_H)\nabla u^h + \beta(u_H)) - \gamma(u_H)\cdot\nabla u_H - g(u_H), b_E\overline{J}_{1,E})_{\omega_E} \\
&\quad = T_1 + T_2 + T_3 + T_4 + T_5,
\end{aligned} \tag{3.25}$$

where and hereafter $\nabla_h \cdot f$ is the function whose restriction to K is $\nabla \cdot f$.

Using Cauchy–Schwarz inequality, (3.15), (3.14) and the method to estimate S_i, $1 \le i \le 4$ in (3.17), we obtain

$$|T_1| \le Ch_E^{-\frac{1}{2}}|u-u^h|_{1,\omega_E}||\overline{J}_{1,E}||_{0,E} + Ch_E^{-\frac{1}{2}}||u-u_H||_{0,\omega_E}||\overline{J}_{1,E}||_{0,E}, \tag{3.26}$$

and

$$|T_2| + |T_3| + |T_4| \le Ch_E^{-\frac{1}{2}}||u-u_H||_{0,\omega_E}||\overline{J}_{1,E}||_{0,E}. \tag{3.27}$$

By Cauchy–Schwarz inequality and (3.14),

$$\begin{aligned}
|T_5| &\le ||\nabla_h\cdot(\alpha(u_H)\nabla u^h + \beta(u_H)) - \gamma(u_H)\cdot\nabla u_H - g(u_H)||_{0,\omega_E}||b_E\overline{J}_{1,E}||_{0,\omega_E} \\
&\le Ch_E^{\frac{1}{2}}||\nabla_h\cdot(\alpha(u_H)\nabla u^h + \beta(u_H)) - \gamma(u_H)\cdot\nabla u_H - g(u_H)||_{0,\omega_E}||\overline{J}_{1,E}||_{0,E}.
\end{aligned} \tag{3.28}$$

Combining (3.22), (3.23), (3.25), (3.26), (3.27) with (3.28) yields

$$h_E^{\frac{1}{2}}||\overline{J}_{1,E}||_{0,E} \leq C||u-u^h||_{1,\omega_E} + C||u-u_H||_{0,\omega_E} + h_E^{\frac{1}{2}}||\overline{J}_{1,E} - J_{1,E}||_{0,E}$$
$$+ Ch_E||\nabla_h \cdot (\alpha(u_H)\nabla u^h + \beta(u_H)) - \gamma(u_H)\cdot\nabla u_H - g(u_H)||_{0,\omega_E}. \tag{3.29}$$

The desired result follows from (3.21), (3.29) and Lemma 13. □

Now, we present the lower bound on the error $u - u^h$ in the H^1-norm of Algorithm 1 for (2.9).

Theorem 2 *Assume that $u \in W^{2,2+\epsilon}(\Omega)$, $\epsilon > 0$, and $u^h \in S_h$ are the solutions of (2.9) and Algorithm 1, respectively. Then, we can choose $H = \mathcal{O}(h^\mu)$, $\mu > 1/2$ such that*

$$\eta_{1,R} + \eta_{1,J} \leq C||u-u^h||_1 + h.o.t.. \tag{3.30}$$

Proof Summing K over $\mathcal{T}_h$ in Lemma 13 and from Lemma 2, we get

$$\begin{aligned}\eta_{1,R}^2 &\leq C||u-u^h||_1^2 + C||u-u_H||^2 + C\mathrm{osc}_{1,R}^2\\ &\leq C||u-u^h||_1^2 + CH^4 + C\mathrm{osc}_{1,R}^2.\end{aligned} \tag{3.31}$$

Similarly, from Lemma 14, we have

$$\begin{aligned}\eta_{1,J}^2 &\leq C||u-u^h||_1^2 + C||u-u_H||^2 + C\mathrm{osc}_1^2\\ &\leq C||u-u^h||_1^2 + CH^4 + C\mathrm{osc}_1^2.\end{aligned} \tag{3.32}$$

For $H = \mathcal{O}(h^\mu)$, $\mu > 1/2$, H^2 is a higher order term compared with h. Since $\mathrm{osc}_{1,R}$ and osc_1 are *h.o.t.*, the desired result follows from (3.31) and (3.32). □

From Theorems 1 and 2, we know that $\eta_{1,R} + \eta_{1,J}$ can be used as the a posteriori error estimator of Algorithm 1 for (2.9).

4 A Posteriori Error Estimates of Algorithm 2

In this section, we develop the computable upper and lower bounds on the error $u - u^h$ in the H^1-norm of Algorithm 2 for (1.1).

4.1 A Reliable Bound on the Error of Algorithm 2

The following lemma plays a key role in the derivation of the upper bound on the error $u-u^h$ in the H^1-norm of Algorithm 2 for (1.1).

Lemma 15 *Assume that $u \in W^{2,4}(\Omega)$ and $u^h \in S_h$ are the solutions of (1.1) and Algorithm 2, respectively. Then for the error $u - u^h$ and $v \in H_0^1(\Omega)$, we have*

$$A'(u; u-u^h, v) = Q_1 + \cdots + Q_5, \tag{4.1}$$

where

$$Q_1 = \sum_{K\in\mathcal{T}_h}\int_K \Big(\nabla\cdot F(u^h,\nabla u^h) - g(u^h,\nabla u^h)\Big)(v - v^I)\mathrm{d}x,$$

$$Q_2 = -\sum_{E\in\mathcal{E}_h^0}\int_E [\![F(u_h,\nabla u_h)]\!](v - v^I)\mathrm{d}s,$$

$$Q_3 = a'(u; u-u^h, v^I) - a'(u_H; u-u^h, v^I),$$

$$Q_4 = a'(u_H; u-u_H, v^I) - a'(u; u-u_H, v^I),$$

$$Q_5 = R(u,u_H,v^I) - R(u,u^h,v^I) + R(u,u^h,v),$$

and v^I is the Scott-Zhang interpolant of v given in Lemma 10.

Proof Since $v, v^I \in H_0^1(\Omega)$, from (2.3) we have $A(u,v) = A(u,v^I) = 0$. Then, it follows from (2.7) that

$$\begin{aligned} A'(u; u-u^h, v) &= A(u,v) - A(u^h,v) + R(u,u^h,v)\\ &= -A(u^h,v) + R(u,u^h,v)\\ &= -A(u^h,v-v^I) + A(u,v^I) - A(u^h,v^I) + R(u,u^h,v)\\ &= -A(u^h,v-v^I) + A'(u;u-u^h,v^I) - R(u,u^h,v^I) + R(u,u^h,v)\\ &= -A(u^h,v-v^I) + \Big[A'(u;u-u^h,v^I) - A'(u_H;u-u^h,v^I)\Big]\\ &\quad + A'(u_H;u-u^h,v^I) - R(u,u^h,v^I) + R(u,u^h,v). \end{aligned} \tag{4.2}$$

From Algorithm 2 and (2.7), we have

$$\begin{aligned} A'(u_H; u-u^h, v^I) &= A'(u_H;u,v^I) - A'(u_H;u^h,v^I)\\ &= A'(u_H;u,v^I) - A'(u_H;u_H,v^I) + A(u_H,v^I)\\ &= A'(u_H;u-u_H,v^I) + A(u_H,v^I)\\ &= A'(u_H;u-u_H,v^I) - A'(u;u-u_H,v^I)\\ &\quad + A'(u;u-u_H,v^I) + A(u_H,v^I)\\ &= \Big[A'(u_H;u-u_H,v^I) - A'(u;u-u_H,v^I)\Big]\\ &\quad + A(u,v^I) + R(u,u_H,v^I)\\ &= \Big[A'(u_H;u-u_H,v^I) - A'(u;u-u_H,v^I)\Big]\\ &\quad + R(u,u_H,v^I). \end{aligned} \tag{4.3}$$

Substituting (4.3) into (4.2) yields

$$\begin{aligned} A'(u;u-u^h,v) = -A(u^h,v-v^I) &+ \Big[A'(u;u-u^h,v^I) - A'(u_H;u-u^h,v^I)\Big]\\ &+ \Big[A'(u_H;u-u_H,v^I) - A'(u;u-u_H,v^I)\Big]\\ &+ \Big[R(u,u_H,v^I) - R(u,u^h,v^I) + R(u,u^h,v)\Big]. \end{aligned} \tag{4.4}$$

Applying Green's formula to the first term on the right-hand side of (4.4) gives

$$\begin{aligned}
&-A(u^h, v - v^I)\\
&\quad = -\sum_{K\in\mathcal{T}_h}\int_K F(u^h,\nabla u^h)\cdot\nabla(v-v^I)\mathrm{d}x - \int_\Omega g(u^h,\nabla u^h)(v-v^I)\mathrm{d}x\\
&\quad = \sum_{K\in\mathcal{T}_h}\int_K \Big(\nabla\cdot F(u^h,\nabla u^h) - g(u^h,\nabla u^h)\Big)(v-v^I)\mathrm{d}x\\
&\qquad - \sum_{E\in\mathcal{E}_h^0}\int_E [\![F(u^h,\nabla u^h)]\!](v-v^I)\mathrm{d}s. \qquad (4.5)
\end{aligned}$$

Combining (4.4) with (4.5) completes the proof. □

Motivated by the above lemma, we introduce locally computable quantities which will be used in the definition of the residual-based a posteriori error estimator of Algorithm 2 for (1.1).

Definition 2 On each element $K \in \mathcal{T}_h$ and each interior edge $E \in \mathcal{E}_h^0$, define the element residual and the edge residual by, respectively,

$$\begin{aligned}
R_{2,K} &= R_{2,K}(u^h) = \nabla\cdot F(u^h,\nabla u^h) - g(u^h,\nabla u^h),\\
J_{2,E} &= J_{2,E}(u^h) = [\![F(u^h,\nabla u^h)]\!]_E
\end{aligned}$$

and define the local error estimators $\eta_{2,R}(K)$ and $\eta_{2,J}(E)$ by

$$\eta_{2,R}(K)^2 = h_K^2||R_{2,K}||_{0,K}^2,\quad \eta_{2,J}(E)^2 = h_E||J_{2,E}||_{0,E}^2.$$

Define the global error estimators by

$$\eta_{2,R} = \left(\sum_{K\in\mathcal{T}_h}\eta_{2,R}(K)^2\right)^{\frac{1}{2}},\quad \eta_{2,J} = \left(\sum_{E\in\mathcal{E}_h^0}\eta_{2,J}(E)^2\right)^{\frac{1}{2}}.$$

We are now in position to develop a reliable estimate for the error $u - u^h$ in the H^1-norm of Algorithm 2 for (1.1). For this purpose, we will estimate the terms on the right-hand side of (4.1) separately.

Theorem 3 *Assume that $u \in W^{2,4}(\Omega)$ and $u^h \in S_h$ are the solutions of (1.1) and Algorithm 2, respectively. Then, we can choose $H = \mathcal{O}(h^\mu)$, where $\mu > 1/2$ if $\delta_2 = 1$; $\mu > 3/8$ if $\delta_2 = 0$ and $\delta_1 = 1$ and $\mu > 1/4$ if $\delta_2 = \delta_1 = 0$ such that*

$$||u - u^h||_1 \le C(\eta_{2,R} + \eta_{2,J}) + h.o.t..$$

Proof The estimation of Q_1 and Q_2 is similar to (3.6) and (3.7), respectively

$$|Q_1| \le C\eta_{2,R}||v||_1,\quad |Q_2| \le C\eta_{2,J}||v||_1. \qquad (4.6)$$

For the third term Q_3, we rewrite it as follows

$$\begin{aligned}
Q_3 &= A'(u; u - u^h, v^I) - A'(u_H; u - u^h, v^I)\\
&= \Big((a(u) - a(u_H))\nabla(u-u^h) + (b(u) - b(u_H))(u-u^h), \nabla v^I\Big)\\
&\quad + \Big((c(u) - c(u_H))\cdot\nabla(u-u^h) + (d(u) - d(u_H))(u-u^h), v^I\Big). \qquad (4.7)
\end{aligned}$$

Using integral form of Taylor's formula, we write

$$\begin{aligned} a(u) - a(u_H) &= D_z F(u, \nabla u) - D_z F(u_H, \nabla u_H) \\ &= D_{zy} \widetilde{F}(u - u_H) + D_{zz} \widetilde{F} \nabla (u - u_H), \end{aligned} \tag{4.8}$$

where

$$D_{zy} \widetilde{F} = \int_0^1 D_{zy} F(v(t))\, \mathrm{d}t, \quad D_{zz} \widetilde{F} = \int_0^1 D_{zz} F(v(t))\, \mathrm{d}t, \tag{4.9}$$

with $v(t) = \big(u_H + t(u - u_H), \nabla(u_H + t(u - u_H))\big)$.

Similarly, we get

$$b(u) - b(u_H) = D_{yy} \widetilde{F}(u - u_H) + D_{yz} \widetilde{F} \nabla (u - u_H). \tag{4.10}$$

$$c(u) - c(u_H) = D_{zy} \widetilde{g}(u - u_H) + D_{zz} \widetilde{g} \nabla (u - u_H), \tag{4.11}$$

$$d(u) - d(u_H) = D_{yy} \widetilde{g}(u - u_H) + D_{yz} \widetilde{g} \nabla (u - u_H), \tag{4.12}$$

where $D_{yy}\widetilde{F}$, $D_{yz}\widetilde{F}$, $D_{zy}\widetilde{g}$, $D_{zz}\widetilde{g}$, $D_{yy}\widetilde{g}$ and $D_{yz}\widetilde{g}$ are defined as $D_{zy}\widetilde{F}$ and $D_{zz}\widetilde{F}$ in (4.9).

Recalling the definitions of δ_2 and δ_1 and noticing that the two terms containing $a(u) - a(u_H)$ and $c(u) - c(u_H)$ in (4.7) will disappear if $\delta_2 = \delta_1 = 0$. Substituting (4.8), (4.10)–(4.12) into (4.7) and applying Hölder's inequality, we get

$$\begin{aligned} |Q_3| &\le C||u - u_H||_{0,4}||u - u^h||_{0,4}||v^I||_1 + C\delta_1||u - u_H||_{1,4}||u - u^h||_{0,4}||v^I||_1 \\ &\quad + C\delta_2||u - u_H||_{1,4}||u - u^h||_{1,4}||v^I||_1. \end{aligned} \tag{4.13}$$

Using Lemmas 2, 9 and 10, we have

$$\begin{aligned} |Q_3| &\le CH^2 \left(h + h^{-\frac{1}{2}} \left(H^4 + \delta_1 H^3 + \delta_2 H^2\right)\right) ||v||_1 \\ &\quad + C(\delta_1 + \delta_2) H \left(h + h^{-\frac{1}{2}} \left(H^4 + \delta_1 H^3 + \delta_2 H^2\right)\right) ||v||_1 \\ &\le Ch^{-\frac{1}{2}} \left(H^6 + \delta_1 H^5 + \delta_2 H^4\right) ||v||_1 \\ &\quad + C(\delta_1 + \delta_2) h^{-\frac{1}{2}} \left(H^5 + \delta_1 H^4 + \delta_2 H^3\right) ||v||_1 + h.o.t.||v||_1. \end{aligned} \tag{4.14}$$

Similar to the estimation of Q_3, we can get

$$\begin{aligned} |Q_4| &\le C||u - u_H||_{0,4}||u - u_H||_{0,4}||v||_1 + C\delta_1||u - u_H||_{1,4}||u - u_H||_{0,4}||v||_1 \\ &\quad + C\delta_2||u - u_H||_{1,4}||u - u_H||_{1,4}||v||_1 \\ &\le C\left(H^4 + \delta_1 H^3 + \delta_2 H^2\right) ||v||_1. \end{aligned} \tag{4.15}$$

Applying (2.8), Lemmas 2, 9 and 10, we can get the estimation of Q_5

$$\begin{aligned} |Q_5| &\le |R(u, u_H, v^I)| + |R(u, u^h, v^I)| + |R(u, u^h, v)| \\ &\le C(||u - u_H||_{0,4}^2 + \delta_1||u - u_h||_{0,4}||u - u_H||_{1,4} + \delta_2||u - u_H||_{1,4}^2)||v||_1 \\ &\quad + C(||u - u^h||_{0,4}^2 + \delta_1||u - u^h||_{0,4}||u - u^h||_{1,4} + \delta_2||u - u^h||_{1,4}^2)||v||_1 \\ &\le C\left(H^4 + \delta_1 H^3 + \delta_2 H^2\right) ||v||_1 + C\left(h^2 + h^{-1}\left(H^8 + \delta_1^2 H^6 + \delta_2^2 H^4\right)\right) ||v||_1 \\ &\le C\left(H^4 + \delta_1 H^3 + \delta_2 H^2\right) ||v||_1 \\ &\quad + Ch^{-1}\left(H^8 + \delta_1^2 H^6 + \delta_2^2 H^4\right) ||v||_1 + h.o.t.||v||_1. \end{aligned} \tag{4.16}$$

From (4.1), (4.6), (4.14)–(4.16), we get

$$\frac{A'(u;u-u^h,v)}{||v||_1} \le C(\eta_{2,R}+\eta_{2,J}) + Ch^{-\frac{1}{2}}\left(H^6+\delta_1 H^5+\delta_2 H^4\right) \\ + C(\delta_1+\delta_2)h^{-\frac{1}{2}}\left(H^5+\delta_1 H^4+\delta_2 H^3\right) \\ + C\left(H^4+\delta_1 H^3+\delta_2 H^2\right) \\ + Ch^{-1}\left(H^8+\delta_1^2 H^6+\delta_2^2 H^4\right) + h.o.t.. \tag{4.17}$$

Choosing $H=\mathcal{O}(h^\mu)$, $\mu>1/2$ if $\delta_2=1$; $\mu>3/8$ if $\delta_2=0,\delta_1=1$; and $\mu>1/4$ if $\delta_2=\delta_1=0$, the second, third, fourth and fifth terms on the right-hand side of (4.17) are *h.o.t.* Then

$$\frac{A'(u,u-u^h,v)}{||v||_1} \le C(\eta_{2,R}+\eta_{2,J}) + h.o.t.. \tag{4.18}$$

The proof is completed from Lemma 4. □

If the solution u of (1.1) has higher smoothness, $u\in W^{2,\infty}(\Omega)$, the choice of H for $\delta_2=0,\delta_1=1$ can be improved as $H=\mathcal{O}(h^\mu)$, $\mu>1/3$.

Theorem 4 *Assume that* $u\in W^{2,\infty}(\Omega)$ *and* $u^h\in S_h$ *are the solutions of* (1.1) *and Algorithm* 2*, respectively. Then, we can choose* $H=\mathcal{O}(h^\mu)$, $\mu>1/3$ *for* $\delta_2=0$ *and* $\delta_1=1$ *such that*

$$||u-u^h||_1 \le C(\eta_{2,R}+\eta_{2,J}) + h.o.t..$$

Proof The estimation of Q_1,Q_2,Q_4,Q_5 are the same as those in the proof of Theorem 3 with a simple modification with $\delta_2=0$. For the term Q_3, we have from (4.7), Hölder's inequality and Lemma 8 that

$$|Q_3| \le C||u-u_H||_1||u-u^h||_{1,\infty}||v^I||_1 \\ \le CH(||u-u_h||_{1,\infty}+||u_h-u^h||_{1,\infty})||v||_1 \\ \le CH(h+H^3|\ln h|)||v||_1 \\ \le CH^4|\ln h|\ ||v||_1 + h.o.t.||v||_1. \tag{4.19}$$

The first term on the right-hand side of (4.19) is also a *h.o.t.* for $H=\mathcal{O}(h^\mu)$, $\mu>1/4$. Noting that for $H=\mathcal{O}(h^\mu)$, $\mu>1/3$, Q_4 and Q_5 are also the *h.o.t.*. Then we have

$$\frac{A'(u,u-u^h,v)}{||v||_1} \le C(\eta_{2,R}+\eta_{2,J}) + h.o.t., \tag{4.20}$$

which completes the proof from Lemma 4. □

4.2 A Lower Bound on the Error of Algorithm 2

In this subsection, we present the lower bound on the error $u-u^h$ in the H^1-norm of Algorithm 2 for (1.1), whose proofs are analogous to those in Sect. 3.2. For sake of completeness, we present the main steps in this subsection.

As in Sect. 3.2, for the residual $R_{2,K}$ and the jump $J_{2,E}$ over the element K and the interior edge E, we define the oscillations $\mathrm{osc}_{2,R}(K)$, $\mathrm{osc}_{2,J}(E)$ and the total oscillation osc_2.

First, for the element residual $R_{2,K}$, we have the following lower bound.

Lemma 16 *Assume that* $u \in W^{2,4}(\Omega)$ *and* $u^h \in S_h$ *are the solutions of* (1.1) *and Algorithm* 2*, respectively. Then, there exists a constant* $C > 0$ *such that*

$$\eta_{2,R}(K)^2 \le C||u-u^h||_{1,K}^2 + C\mathrm{osc}_{2,R}(K)^2, \quad \forall K \in \mathcal{T}_h. \tag{4.21}$$

Proof Similar to (3.16), we have

$$h_K||R_{2,K}||_{0,K} \le h_K||\overline{R}_{2,K}||_{0,K} + \mathrm{osc}_{2,R}(K). \tag{4.22}$$

Thus, we only estimate $h_K||\overline{R}_{2,K}||_{0,K}$ in the following. Similar to (3.17), we have

$$\begin{aligned}\frac{9}{20}||\overline{R}_{2,K}||_{0,K}^2 &= (F(u,\nabla u) - F(u^h,\nabla u^h), \nabla(b_K\overline{R}_{2,K}))_K \\ &\quad + (g(u,\nabla u) - g(u^h,\nabla u^h), b_K\overline{R}_{2,K})_K \\ &\quad - (R_K - \overline{R}_{2,K}, b_K\overline{R}_{2,K})_K \\ &= R_1 + R_2 + R_3.\end{aligned} \tag{4.23}$$

We note that by integral form of the Taylor's formula, we write

$$F(u,\nabla u) - F(u^h,\nabla u^h) = \widetilde{F}_y\,(u-u^h) + \widetilde{F}_z\,\nabla(u-u^h), \tag{4.24}$$

where

$$\widetilde{F}_y = \int_0^1 D_yF(v(t))\,\mathrm{d}t, \quad \widetilde{F}_z = \int_0^1 D_zF(v(t))\,\mathrm{d}t, \tag{4.25}$$

with $v(t) = \big(u_h + t(u-u^h), \nabla(u_h + t(u-u^h))\big)$, and D_yF and D_zF are the Jacobian of $F(y,z)$. Using this representation, appealing to (2.1), Cauchy–Schwarz inequality and (3.13), we get

$$\begin{aligned}|R_1| &\le C\|u-u^h\|_{1,K}\|\nabla(b_K\overline{R}_{2,K})\|_{0,K} \\ &\le Ch_K^{-1}\|u-u^h\|_{1,K}||\overline{R}_{2,K}||_{0,K}.\end{aligned} \tag{4.26}$$

Using similar representation for $g(y,z)$, we estimate

$$|R_2| \le C\|u-u^h\|_{1,K}\|b_K\overline{R}_{2,K}\|_{0,K} \le C\|u-u^h\|_{1,K}\|\overline{R}_{2,K}\|_{0,K}. \tag{4.27}$$

Using Cauchy–Schwarz inequality and $\max_{x\in K} b(x) = 1$, we get

$$|R_3| \le ||R_{2,K} - \overline{R}_{2,K}||_{0,K}||\overline{R}_{2,K}||_{0,K}. \tag{4.28}$$

Combining (4.23), (4.26), (4.27) with (4.28) yields

$$h_K||\overline{R}_{2,K}||_{0,K} \le C||u-u^h||_{1,K} + Ch_K||R_{2,K} - \overline{R}_{2,K}||_{0,K}. \tag{4.29}$$

It follows from (4.29) and (4.22) that

$$h_K||R_{2,K}||_{0,K} \le C||u-u^h||_{1,K} + C\mathrm{osc}_{2,R}(K). \tag{4.30}$$

Squaring on both sides of (4.30) completes the proof. □

Lemma 17 *Assume that* $u \in W^{2,4}(\Omega)$ *and* $u^h \in S_h$ *are the solutions of* (1.1) *and Algorithm* 2*, respectively. Then, there exists a constant* $C > 0$ *such that for* $E = \partial K \cap \partial K'$

$$\eta_{2,J}(E)^2 \le C||u-u^h||_{1,\omega_E}^2 + C\mathrm{osc}_{2,R}(\omega_E)^2 + C\mathrm{osc}_{2,J}(E)^2, \tag{4.31}$$

where $\omega_E = K \cup K'$, $\mathrm{osc}_{2,R}(\omega_E)^2 = \mathrm{osc}_{2,R}(K)^2 + \mathrm{osc}_{2,R}(K')^2$.

Proof Similar to (3.21), (3.22), (3.23) and (3.25), we have

$$h_E^{\frac{1}{2}}||J_{2,E}||_{0,E} \leq h_E^{\frac{1}{2}}||\overline{J}_{2,E}||_{0,E} + \mathrm{osc}_{2,J}(E), \tag{4.32}$$

$$\begin{aligned} \frac{2}{3}||\overline{J}_{2,E}||_{0,E}^2 &= (J_{2,E}, b_E\overline{J}_{2,E})_E + (\overline{J}_{2,E} - J_{2,E}, b_E\overline{J}_{2,E})_E \\ &\leq (J_{2,E}, b_E\overline{J}_{2,E})_E + ||\overline{J}_{2,E} - J_{2,E}||_{0,E}||\overline{J}_{2,E}||_{0,E}, \end{aligned} \tag{4.33}$$

and

$$\begin{aligned} (J_{2,E}, b_E\overline{J}_{2,E})_E &= (F(u^h, \nabla u^h) - F(u, \nabla u), \nabla(b_E\overline{J}_{2,E}))_{\omega_E} \\ &\quad + (g(u^h, \nabla u^h) - g(u, \nabla u), b_E\overline{J}_{2,E})_{\omega_E} \\ &\quad + (\nabla_h \cdot F(u^h, \nabla u^h) - g(u^h, \nabla u^h), b_E\overline{J}_{2,E})_{\omega_E} \\ &= R_1 + R_2 + R_3. \end{aligned} \tag{4.34}$$

Using (4.24), Cauchy–Schwarz inequality and (3.15), we obtain

$$|R_1| \leq Ch_E^{-\frac{1}{2}}|u - u^h|_{1,\omega_E}||\overline{J}_{2,E}||_{0,E}. \tag{4.35}$$

By Cauchy–Schwarz inequality and (3.14), we have

$$|R_2| \leq Ch_E^{\frac{1}{2}}||u - u^h||_{1,\omega_E}||\overline{J}_{2,E}||_{0,E}. \tag{4.36}$$

and

$$|R_3| \leq Ch_E^{\frac{1}{2}}||\nabla_h \cdot F(u^h, \nabla u^h) - g(u^h, \nabla u^h)||_{0,\omega_E}||\overline{J}_{2,E}||_{0,E}. \tag{4.37}$$

Combining (4.33)–(4.36) with (4.37) yields

$$\begin{aligned} h_E^{\frac{1}{2}}||\overline{J}_{2,E}||_{0,E} &\leq C||u - u^h||_{1,\omega_E} + Ch_E||\nabla_h \cdot F(u^h, \nabla u^h) - g(u^h, \nabla u^h)||_{0,\omega_E} \\ &\quad + h_E^{\frac{1}{2}}||\overline{J}_{2,E} - J_{2,E}||_{0,E}. \end{aligned} \tag{4.38}$$

Applying Lemma 16, from (4.32) and (4.38), we can obtain the desired result. □

Based on Lemmas 16 and 17, we immediately get the following theorem, whose proof is similar to that of Theorem 2 and we omit it here.

Theorem 5 *Assume that* $u \in W^{2,4}(\Omega)$ *and* $u^h \in S_h$ *are the solutions of* (1.1) *and Algorithm* 2*, respectively. Then, we have*

$$\eta_{2,R} + \eta_{2,J} \leq C||u - u^h||_1 + h.o.t..$$

From Theorems 3 and 5, we know that $\eta_{2,R} + \eta_{2,J}$ can be used as the a posteriori error estimator of Algorithm 2 for (1.1).

5 Numerical Experiments

In this section, we present three numerical examples to illustrate the performance of the error estimators that have been analyzed earlier. Our focus is to observe the ability of the error estimates to imitate the convergence behavior of the exact errors in the H^1-norm.

Table 1 A posteriori estimates of the Algorithm 1 for the first example

H	h	$\|u-u_h\|_1$	$\eta_{1,R}$	$\eta_{1,J}$	η_1	R_1
9.0909e−2	1.0000e−2	2.9964e−3	1.2723e−3	3.5015e−3	4.7739e−3	0.62
7.6923e−2	6.9444e−3	2.0105e−3	8.3588e−4	2.4200e−3	3.2559e−3	0.61
6.6667e−2	5.1020e−3	1.4510e−3	5.9635e−4	1.7805e−3	2.3769e−3	0.61
5.8824e−2	3.9063e−3	1.1140e−3	4.4967e−4	1.3699e−3	1.8195e−3	0.61
5.2632e−2	3.0864e−3	8.5476e−4	3.5008e−4	1.0789e−3	1.4290e−3	0.59
4.7619e−2	2.5000e−3	6.9049e−4	2.7954e−4	8.6876e−4	1.1483e−3	0.60

In all examples, we use the same true solution

$$u(x_1, x_2) = x_1^3 \ln(x_1) x_2^3 \ln(x_2),$$

designate the expression of $F(x, u, \nabla u)$, and set the function $g(x, u)$ to satisfy the partial differential Eq. (1.1). For the following three examples, the nonlinear elliptic problems are posed in a domain $\Omega = (0, 1) \times (0, 1)$, which is partitioned into triangles, resulting a quasi-uniform mesh with size h. Based on these meshes, the conforming linear finite element space S_h are constructed.

To solve the nonlinear systems of algebra equations generating by the conforming linear finite element approximation on the coarse meshes in Algorithms 1 and 2, we shall use a simple fixed point type iteration method, that is, to write the weak form as to find $u_h^{(k)} \in S_h$ such that

$$\left(A\left(x, u_h^{(k-1)}, \nabla u_h^{(k-1)}\right) \nabla u_h^{(k)}, \nabla v_h\right) + \left(B\left(x, u_h^{(k-1)}\right) u, v_h\right) = (f, v_h), \quad \forall v_h \in S_h,$$

where $u_h^{(k-1)}$ is the solution obtained in the previous iteration step. We remark that this type of linearization scheme deals well with the following three examples.

For the first example, we consider the problem (2.9), a special case of (1.1) with

$$F(x, u, \nabla u) = (1 - 0.9 \sin(20\pi u)) \nabla u.$$

The function $g(x, u)$ is independent of u and chosen such that $g(x) = \nabla \cdot F(x, u, \nabla u)$.

We use Algorithm 1 to solve this problem. The numerical results are shown in Table 1, where the comparison of the error estimator η_i against the exact error in H^1-norm is denote by

$$R_i = \frac{\|u - u^h\|_1}{\eta_i}, \quad \eta_i = \eta_{i,R} + \eta_{i,J}, \quad i = 1, 2.$$

Note that the relation $H \approx O(h^r)$, $r > 1/2$ holds. From this table, we can see that the estimator η_1 exhibits the similar convergence behavior as the exact errors in H^1-norm.

For the second example, we consider the equation of prescribed mean curvature described in Sect. 2.1. We use Algorithm 2 to solve this problem. The same meshes are used as those in the first example. The numerical results are shown in Table 2, from which we can see the same convergence rate of the a posterior error estimator η_2 and the exact error in H^1-norm.

For the third example, we consider the following semilinear problem

$$\begin{cases} -\Delta u + u^3 = f, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega. \end{cases}$$

Table 2 A posteriori estimates of the Algorithm 2 for the second example

H	h	$\|u-u_h\|_1$	$\eta_{2,R}$	$\eta_{2,J}$	η_2	R_2
9.0909e−2	1.0000e−2	1.0919e−3	3.0629e−3	5.5813e−3	8.6442e−3	0.13
7.6923e−2	6.9444e−3	6.9382e−4	1.9285e−3	3.5597e−3	5.4882e−3	0.13
6.6667e−2	5.1020e−3	3.5202e−4	9.8019e−4	1.8137e−3	2.7939e−3	0.13
5.8824e−2	3.9063e−3	2.7134e−4	7.5263e−4	1.3934e−3	2.1460e−3	0.13
5.2632e−2	3.0864e−3	2.1578e−4	5.9323e−4	1.1003e−3	1.6935e−3	0.13
4.7619e−2	2.5000e−3	1.7546e−4	4.7739e−4	8.8642e−4	1.3638e−3	0.13

Table 3 A posteriori estimates of the Algorithm 2 for the third example

H	h	$\|u-u_h\|_1$	$\eta_{2,R}$	$\eta_{2,J}$	η_2	R_2
2.1544e−1	1.0000e−2	6.9276e−4	2.3176e−4	3.5720e−3	3.8038e−3	0.18
1.9079e−1	6.9444e−3	4.7595e−4	1.6016e−4	2.4703e−3	2.6304e−3	0.18
1.7215e−1	5.1020e−3	3.4960e−4	1.1801e−4	1.8196e−3	1.9376e−3	0.18
1.5749e−1	3.9063e−3	2.6791e−4	9.0357e−5	1.3979e−3	1.4882e−3	0.18
1.4560e−1	3.0864e−3	2.1127e−4	7.1499e−5	1.1035e−3	1.1750e−3	0.18
1.3572e−1	2.5000e−3	1.6977e−4	5.7573e−5	8.8899e−4	9.4657e−4	0.18

We use Algorithm 2 to solve this problem, and the meshes are generated with $h \approx H^3$. The numerical results are shown in Table 3, which also confirm our theoretical results.

6 Summary and Concluding Remarks

In this article we established the a posteriori error estimates of two-grid Algorithm 2 and Algorithm 1 (more simple than Algorithm 2), respectively, for the nonlinear elliptic problems (1.1) and (2.9), a special case of (1.1). By choosing the coarse mesh-size appropriately, we derived the global upper and lower bounds on the error $u - u^h$ in the H^1-norm. Numerical experiments are also provided to illustrate the performance of the proposed estimators.

Although we derived the global upper and lower bounds on the error $u - u^h$ in the H^1-norm for (2.9) only for $H = \mathcal{O}(h^\mu)$, $\mu > 1/2$, in the numerical experiments we found η can also be used as the error indicator of $||u - u^h||_1$ for $H = \mathcal{O}(h^{1/2})$. Similar numerical results were observed for (1.1) for $H = \mathcal{O}(h^\mu)$, $\mu = 1/2$ if $\delta_2 = 1$; $\mu = 1/3$ if $\delta_2 = 0$ and $\delta_1 = 1$ and $\mu = 1/4$ if $\delta_2 = \delta_1 = 0$.

The results in this article can be easily extended to the conforming finite element space of polynomials of degree $r \geq 2$. Moreover, it is not difficult to extend our analysis to the nonlinear problems with Neumann boundary condition.

In this article, we study the a posteriori error estimates of the two-grid finite element method for (1.1) under the assumption $u \in H_0^1(\Omega) \cap W^{2,2+\epsilon}(\Omega)$ for some $\epsilon > 0$, which has been used to develop the existence and uniqueness of the finite element approximation of (1.1) in [43]. The a posteriori error estimates of finite element method for the nonlinear elliptic problems with lower regularity deserve further study in the future.

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